

Caustic singularity in Hořava-Lifshitz gravity

M.R.Setare^{1,*} and D. Momeni^{2,†}

¹*Department of Campus of Bijar , University of Kurdistan, Bijar, IRAN.*

²*Department of Physics, Faculty of Sciences,
Tarbiat Moa'llem University, Tehran, Iran*

Abstract

In this note we searched for a family of solutions with Caustic singularity in non relativistic-renormalizable Hořava-Lifshitz (HL) theory without the general covariant. We show that in infrared (IR) limit and with a deviation from $\lambda = 1$ we have no caustic singularity. Also in ultraviolet (UV) regime and for Ricci flat 3-dimensional ($3d$) spaces and codimension 1 and for $\lambda \neq 1$ the non linear terms should help bouncing this kind of most dangerous would be caustics. But if $3d$ curvature does not vanish, higher curvature terms do help caustics even in codimension one. Thus the arguments in [JCAP 0909:005,2009] are satisfied correctly.

*Electronic address: rezakord@mail.ipm.ir

†Electronic address: d.momeni@yahoo.com

I. INTRODUCTION

HL theory [1, 2] is a non relativistic foliation preserving homeomorphism invariant theory which is stochastic quantized in UV region and in IR limit it mimics the general relativity (GR) with a dark matter [4, 7]. In the other word the gauge symmetries of the system are foliation-preserving diffeomorphisms of spacetime. Also the higher curvature terms in the action can be treated as a generalized modified gravity. They lead to the regular solutions in the UV regime and also make the flatness problem milder [8]. The anisotropic scaling of this model in $z = 3$ critical point solves the horizon problem and also describes the scale invariant cosmological perturbations without any need to the inflation [9, 10]. As a field theoretic model, HL theory is power counting renormalizable in spite of the GR which is not power counting. This is one of the most difficulties of the quantum gravity. If the critical exponent z is fixed in $z = 3$, the amplitude of quantum fluctuations of the scalar field does not change as the energy scale of the system changes. In this case the non linear interactions of graviton are power counting renormalizable and for such values of $z > 3$ is super renormalizable [11]. The power counting renormalizability is achieved by violating the Lorentz invariance with working by anisotropic scaling of time and spatial coordinates. Unlike the GR in HL theory if we want to preserve the foliation preserving diffeomorphism invariance theory must be parity invariance. The original HL model was parity violating. It was subsequent work by Sotiriou et al. [12] that showed that a parity respecting variant of the original HL model could be constructed. It is not the unique option. More recently Horava and Melby-Thompson proposed a new version [13]. In this version the extended gauge symmetry eliminates the scalar graviton and consequently limited the value of the coupling constant λ . Outset we show that if we take a general plane symmetric background for propagation of fields, the IR limit of equation of motions register the familiar $1 + 1$ dimensional wave equation. Secondly this equation designates a square term proportions to the Cosmological constant. Thus we can treat Λ in IR limit as a key for repulsive attraction. Further we show that there is no possibility for Caustic singularity formation, i.e. a singular 3-dimensional Ricci flat extrinsic curvature solution not by higher order terms nor in IR limit. But in UV if $3d$ curvature does not vanish, higher curvature terms do help caustics even in codimension one.

II. IR LIMIT OF HL GRAVITY

Following the Mukohyama[4], at now we know that the HL theory mimics GR plus dark matter[3]. Also between four different versions of this theory only one with projectable condition and without detailed balance (or with a small deviation from this principle) is guaranteed. The general action for HL theory with these conditions reads as

$$I_{IR} = I_{Kin} + I_{z=1} + I_{z=0} = \frac{M_{Pl}^2}{2} \int N dt \sqrt{g} d^3 \vec{x} (K^{ij} K_{ij} - \lambda K^2 + R - 2\Lambda) \quad (1)$$

with an ADM metric [14] in 3+1 decomposition formalism

$$ds^2 = -N(t)^2 dt^2 + g_{ij} dx^i dx^j \quad (2)$$

In order to host instability, thus λ must be either larger than 1 or smaller than 1/3. We know that λ runs from $+\infty$ in the UV to $1+0$ in the IR. According to a phenomenological constraint on properties of the renormalization group (RG), λ must be sufficiently close to 1 at low energy, while $\lambda - 1$ can be of $O(1)$ at larger at high energy. In IR limit or macroscopic objects neglecting from the higher spatial derivatives terms $I_{z=3}, I_{z=2}$ in the action (1) is the single option. The lapse function $N(t)$ is required to be independent of spatial coordinates by the profitability condition. Hence by a space-independent time parameterizations we set the lapse to unity.

A. Equations of motion

By variation of the action (1) in the absence of any matter field with respect to the lapse function $N(t)$, we obtain the Hamiltonian constraint which is global constraint and not the local constraint one's

$$H_g = -\frac{\delta I_g}{\delta N} = \int d^3 \vec{x} \mathcal{H}_g = \frac{M_{Pl}^2}{2} \sqrt{g} (K^{ij} p_{ij} - \Lambda - R - I_{z1}) \quad (3)$$

$$p_{ij} = K_{ij} - \lambda K g_{ij}$$

For shift variation i.e $N^i(t, \vec{x})$ we have the next divergence's like equation for reduced extrinsic curvature tensor p_{ij}

$$-M_{Pl}^2 \sqrt{g} \nabla^i p_{ij} = 0 \quad (4)$$

and finally the 3d metric must obeys from the following equation of motion

$$\begin{aligned} \epsilon_{gij} = g_{ik}g_{jl}\left(\frac{2}{N\sqrt{g}}\frac{\delta I_g}{\delta g_{kl}}\right) = M_{Pl}^2\left[-\frac{1}{N}(\partial_t - N^k\nabla_k)p_{ij} \right. \\ \left. + \frac{1}{N}(p_{ik}\nabla_j N^k + p_{jk}\nabla_i N^k) - Kp_{ij} + 2K_i^k p_{kj} + \frac{1}{2}g_{ij}K^{kl}p_{kl} \right. \\ \left. + \frac{1}{2}\Lambda g_{ij} - G_{ij}\right] + \epsilon_{z>1,ij} = 0 \end{aligned} \quad (5)$$

III. METRIC

We shall restrict ourselves to situations where the space time has plane symmetry. We adopted a coordinates system (t, x, y, z) with plane symmetry such that for them the projectable case satisfied by setting $N(t) = 1$ and the 3 spatial metric g_{ij} be in the form

$$g_{ij} = \text{diag}(1, e^{u(x,t)}, e^{u(x,t)}) \quad (6)$$

and the coordinates $i, j = x, y, z$. That is, the problem is invariant under transformations of the form

$$\begin{aligned} y &\rightarrow y + a \\ z &\rightarrow z + b \\ & , \\ y &\rightarrow y \cos(\theta) + z \sin(\theta) \\ z &\rightarrow z \cos(\theta) - y \sin(\theta) \end{aligned}$$

The four dimensional metric of a spacetime admitting plane symmetry in general may be written as [5](we take $c = 1$)

$$ds^2 = -e^{2F}dt^2 + e^{2G}dx^2 + e^{2H}(dy^2 + dz^2)$$

Where F, G and H are functions of x and t alone. Static and non static exact solutions for HL gravity has been investigated by the authors in Ref. [6].

For future uses we write the Ricci scalar for 3d part of (6) as ¹

$$R = 2u_{xx} + \frac{3}{2}u_x^2 \quad (7)$$

¹ $f_a = \frac{\partial f}{\partial x^a}, \dot{u} = \frac{\partial u}{\partial t}$

This form of metric is planar and stationary thus it is a good example for testing the Mukohyama's idea about the no existence of the vacuum caustic singularities in HL theory. We are searching for at least one 3d Ricci flat ($R = 0$) solution for metric ansatz (6) in such a way that its extrinsic curvature $K = u_t$ diverges. The general form of Ricci flat metric's function may be written as the following

$$u(x, t) = \frac{4}{3} \log\left(\frac{3}{4}(f_1(t)x + f_2(t))\right) \quad (8)$$

Or the following equivalent form

$$u(x, t) = \frac{4}{3} \log(f_1(t)) + \frac{4}{3} \log(x + f_3(t)) + c \quad (9)$$

Where in it $c = (4/3) \ln(3/4)$. If we define $f_4(t)$ by $f_1(t) = (4/3)f_4(t)$ then (9) becomes

$$u(x, t) = \frac{4}{3} \log(f_4(t)) + \frac{4}{3} \log(x + f_3(t))$$

So without any loss of generality, the constant c can be eliminated completely. The unknown functions $f_{i(=1,3)}(t)$ must be determined via substituting in field equations.

IV. EXACT SOLUTION FOR IR LIMIT

In this section we search for all exact solutions for field equations (4), (5) with metric ansatz (6). We divided exact solutions to 2 different class: One with ($R=0$) and another without this constraint. In first case we will show that the Caustic singularity avoids in a natural sense. But in the another case the field equations reduces to a (1+1) dimensional wave equation with an effective sound speed with a wide class of solutions. In this class we can treat Λ as a repulsive force.

A. Ricci flat solution for field equations in IR limit

The field equations (4), (5) after variation action (1) for lapse, shift and 2 independent (but symmetric) components of the metric lead to the next set of partial differential equations

$$-(1 - 2\lambda)\dot{u}^2 + \Lambda + 2u_{xx} + \frac{3}{2}u_x^2 = 0 \quad (10)$$

$$\lambda\ddot{u} + \frac{1}{4}(1 + 2\lambda)\dot{u}^2 + \frac{1}{4}u_x^2 + \frac{1}{2}\Lambda = 0 \quad (11)$$

$$-(1 - 2\lambda)\ddot{u} + \Lambda + u_{xx} + \frac{u_x^2}{2} = 0 \quad (12)$$

We substitute (9) in (10)-(12). In this case (10) gives us

$$\dot{u} = \pm \sqrt{\frac{\Lambda}{1-2\lambda}} \quad (13)$$

Since $\Lambda > 0$ to avoidance from a pure imaginary solution we must have

$$\lambda \leq \frac{1}{2} \quad (14)$$

If the equality sign holds we have a singularity in u_t and consequently from it $K = \infty$. But we remove this special case since we want to keep the value of λ near 1 and also as we will show in next lines we must set $\Lambda = 0$ for consistency with two remaining equations (11), (12). Integrating from (13) using (9) we have

$$f_1(t) = f_1(0) \exp\left(\pm \frac{3}{4} \sqrt{\frac{\Lambda}{1-2\lambda}} t\right) \quad (15)$$

Also for satisfying (11), (12) we must fix

$$f_3(t) = 0 \quad (16)$$

Finally comparing with (11), (12) we obtain the next solution under constraint $\Lambda = 0$. Indeed by adding (11), (12) we have

$$(3\lambda - 1)\ddot{u} + \frac{3}{2}\Lambda + \frac{1}{4}(1 + 2\lambda)\dot{u}^2 = 0 \quad (17)$$

Substituting (9) with (15,16) in (17) we arrive to

$$\text{either } \Lambda = 0 \quad \text{or} \quad \lambda = \frac{7}{10}. \quad (18)$$

Remember that $\lambda = \frac{7}{10}$ is in contradiction with (14). Thus the only possible choose is $\Lambda = 0$. Consequently from (9) we have

$$u(x, t) = \frac{4}{3} \log(x) + c' \quad (19)$$

Which is obviously a no caustic solution, since $K = u_t = 0$ and does not diverge at all. Equation (19) does not represent a caustic but it is certainly odd (in many ways its worse than a caustic). After a shift in coordinates to eliminate c' , this corresponds to the 3-metric

$$g_{ij} = \text{diag}(1, x^{\frac{4}{3}}, x^{\frac{4}{3}})$$

Thus the spatial metric exhibits a naked singularity.

B. $R \neq 0$ solution for field equations in IR limit

In this section we investigate the case with non vanishing Ricci curvature. By elimination of the \dot{u}, u_x from equations (10)-(12) which makes them non linear we relieve

$$\frac{u_{tt}}{c_{eff}^2} = u_{xx} + \frac{2\Lambda(2\lambda + 1)}{2\lambda - 3} \quad (20)$$

where in it

$$c_{eff}^2 = \frac{-5 + 8\lambda + 4\lambda^2 - 4\Lambda + 8\lambda\Lambda}{2\lambda - 3} \quad (21)$$

since we want that the potential function $u(x, t)$ be stable and with a real sound speed, we must have

$$\frac{-5 + 8\lambda + 4\lambda^2 - 4\Lambda + 8\lambda\Lambda}{2\lambda - 3} > 0 \quad (22)$$

or equivalently

$$\text{either } \lambda < \frac{1}{2} \quad \text{or} \quad \lambda > \frac{3}{2}. \quad (23)$$

thus the general form of the metric function $u(x, t)$ is nothing but the familiar d'Alembert solution

$$u(x, t) = H(x \pm c_{eff}t) - \frac{\Lambda(2\lambda + 1)}{2\lambda - 3}x^2 \quad (24)$$

The corresponding spatial metric (introducing an arbitrary function $H(\xi)$ and a distance sacle a) is

$$g_{ij} = \text{diag}(1, H(x \pm c_{eff}t)e^{-ax^2}, H(x \pm c_{eff}t)e^{-ax^2})$$

This is an arbitrary moving pulse of geometry on a Gaussian background. We mention here that the cosmological constant term treats as a forcing term and if we suppose that the metric functions been bound functions in some fixed points in the gravitational part of the manifold, then it seems that at an instant of time calling it $t = 0$ the system is under some initial force. Since we hope that the wavy solutions of the gravitational fields far from it's origin treat as classical GR, thus we can say that in HL theory the cosmological constant term behaves as an external fixed force field.

V. SEARCHING FOR CAUSTIC SINGULARITY IN UV REGIME AND $\lambda \neq 1$

In this section we examine the existence of caustic singularity beyond the IR limit, so we add a non linear term to action (1). For $z = 3$ the minimal coupling must be as a quadratic term of curvature. Thus we write the full modified action as

$$I = \frac{M_{Pl}^2}{2} \int N dt \sqrt{g} d^3 \vec{x} (K^{ij} K_{ij} - \lambda K^2 + R - 2\Lambda + \xi R^2) \quad (25)$$

If we find the exact solutions for metric function $\partial_t u(x, t)$ does not diverge, then we determine that the non linear terms success in stoppage of formation such singularities. For ansatz (6) and action (25), it is easy to show that the reduced equation of motion for metric function $u(x, t)$ is

$$\left(\frac{1}{2} + \lambda\right) u_t^2 + u_{xx}(1 - 6\xi u_{xx} + \frac{19}{2}\xi u_x^2) + u_x^2 \left(\frac{1}{2} - \frac{3}{4}\xi u_x^2\right) - 8\xi u_x u_{xxx} - 8\xi u_{xxxx} = u_{tt} \quad (26)$$

This equation is similar to a Hamilton-Jacobi equation. Then the general solution can be written as an additive function

$$u(x, t) = f(x) + h(t) \quad (27)$$

Since we simultaneously must have the Ricci flat condition (9) thus the only possible consistent form is

$$u(x, t) = \frac{4}{3} \log(f_1(t)) + \frac{4}{3} \log(x) + c \quad (28)$$

Substituting this ansatz in (26) we obtain

$$h_{tt} = \frac{C}{1 - 2\lambda} + \frac{(2\lambda - 1)h_t^2 - 4\Lambda}{2(1 - 2\lambda)}, \quad (29)$$

where C is insignificance constant. Remember that $K = u_t = h_t$ and we are searching for such solution that $k \rightarrow \infty$. The general solution for (29) is nothing else as an elementary function:

$$h(t) = -\frac{t\sqrt{4\Lambda - 2C}}{\sqrt{2\lambda - 1}} - b + 2 \log(-c_2 + c_1 e^{\frac{t\sqrt{4\Lambda - 2C}}{\sqrt{2\lambda - 1}}}) \quad (30)$$

(C, c_1, c_2 all are constant as a function of λ, Λ). The extrinsic curvature is

$$K = \sqrt{\frac{4\Lambda - 2C}{2\lambda - 1}} \frac{c_2 + c_1 e^{\frac{t\sqrt{4\Lambda - 2C}}{\sqrt{2\lambda - 1}}}}{-c_2 + c_1 e^{\frac{t\sqrt{4\Lambda - 2C}}{\sqrt{2\lambda - 1}}}} \quad (31)$$

Obviously in contrast of the former Mukohyama's hypothesis, it seems that this exact solution stricken to the caustic singularity. This would diverges as

$$t \rightarrow t_c = \sqrt{\frac{2\lambda - 1}{4\Lambda - 2C}} \ln\left(\frac{c_2}{c_1}\right) \quad (32)$$

But the x dependence part of (28) i.e $\frac{4}{3} \log(x) + c$ does not satisfy the related x equation (as we will show). Thus this solution which causes a harmful caustic singularity is not acceptable. Thus although in UV regime this kind of singularity is absent at least in order of R^2 action. Indeed (26) with (27) ansatz leads to the following non linear fourth order differential equation for $f(x)$:

$$f_{xxxx} + \frac{1}{32\xi}(-38\xi f_{xx}f_x^2 - 4f_{xx} - 2f_x^2 + 24\xi f_{xx}^2 + 3\xi f_x^4 + 32\xi f_x f_{xxx}) = 0 \quad (33)$$

As we know [15] this is a 4th order ordinary differential equation (ODE) that is missing the dependent variable x . The order can be reduced by introducing a new variable $p(x) = y_x$. If the reduced ODE can be solved for $p(x)$, the solution to the original ODE is determined as a quadrature. The transformation

$$y_x = p(x), y_{xx} = p(x)p(x)_x, \dots \quad (34)$$

yields a reduction of order. If the reduced ODE can be solved for $p(y)$, the solution to the original ODE can be given implicitly.

It is so easy to show that the function $\frac{4}{3} \log(x) + c$ is not a solution for this ODE. Thus the caustic singularity is not formed in UV regime as IR limit. This would not be caustic, the system does not remains in the IR regime insofar as the λ should deviate from 1 by RG flow. These simple calculations prove that there is no caustic singularity in plane symmetry even when the non linear curvature terms insert in the action. Consequently the former Mukohyama's conjecture [3] about the avoidance of caustic singularity in HL theory is true.

VI. CONCLUSION

In this short letter we show that the Mukohyama's conjecture about the avoidance of caustic singularity in HL theory is true. First we showed that in IR limit, there is a family of exact solutions with no caustic singularity and further we showed that the non Ricci flat solutions define an effective sound speed. The stability test of this solution under perturbations show that there is a bound for the value of λ . Secondly we show that even in non linear

regime, i.e. UV region the caustic singularity does not produce in a satisfactory scheme. That solution which causes such singularity does not satisfy fields equations correctly and completely. Thus our work may be an adhoc analytic proof for former Mukohyama's conjecture about the avoidance of caustic singularity in HL theory.

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- [1] Petr Horava, Phys.Rev.D79:084008,2009,[arXiv:0901.3775[hep-th]].
 - [2] Petr Horava, JHEP 0903:020,2009,[arXiv:0812.4287[hep-th]].
 - [3] Shinji Mukohyama, JCAP 0909:005,2009 ,[arXiv:0906.5069 [hep-th]].
 - [4] Shinji Mukohyama, Class.Quant.Grav.27:223101,2010,[arXiv:1007.5199 [hep-th]].
 - [5] A. H. Taub, Ann. Math. 53, 472 (1951)
 - [6] M. R. Setare , D. Momeni, International Journal of Modern Physics D Vol. 19, No. 13 (2010) 2079 – 2094[arXiv:0911.1877v3 [hep-th]]
 - [7] Shinji Mukohyama, Phys. Rev.D 80,064005[arXiv:0905.3563[hep-th]].
 - [8] Gianluca Calcagni, JHEP 0909:112,2009,[arXiv:0904.0829[hep-th]].
 - [9] Shinji Mukohyama, JCAP 0906,001,2009 ,[arXiv:0904.2190 [hep-th]].
 - [10] Elias Kiritsis , Georgios Kofinas , Nucl.Phys.B821:467-480,2009,[arXiv:0904.1334[hep-th]].
 - [11] Rong-Gen Cai, Yan Liu, Ya-Wen Sun, JHEP 0906:010,2009, [arXiv:0904.4104[hep-th]].
 - [12] T. P. Sotiriou, M. Visser and S. Weinfurtner, J. High Energy Phys. 090 (2009) 033.
 - [13] Petr Horava and Charles Melby-Thompson, Phys.Rev.D82:064027,2010,[arXiv:1007.2410v2 [hep-th]].
 - [14] R. Arnowitt, S. Deser, C. W. Misner, Gen. Rel. Gra (2008) 40:1997.
 - [15] George Moseley Murphy, "Ordinary Differential Equations and their Solutions", 1960, sections B2(1,2), and C2(1,2).